Linearized Inviscid-Flow Theory of Two-Dimensional Thin Jet Penetration into a Stream

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The potential flow of a stream that interacts with a two-dimensional thin jet of a different total head, being injected into the stream from an infinite plane surface at an arbitrary angle, is analyzed using natural coordinates. The velocity magnitudes along the interface and the nondimensional shape of the interface between the jet and the stream are obtained as functions of the injection angle and the ratio of the freestream velocity to the velocity in the jet at infinity downstream. Results are presented for several cases when the jet issues at oblique angles from the surface, and also the limiting case when the jet opposes the freestream. The latter case corresponds to the flow due to one branch of a translating two-dimensional jet after the jet has been split into two branches by impingement on the ground. It might also correspond to the flow of a two-dimensional thrust reverser with 180° flow reversal. The calculations show a deeper jet penetration than indicated by previous theories and by a single set of experimental results.

Nomenclature

A_n = coefficients in series expansion
p = static pressure
q = magnitude of the velocity vector at any point in the stream
R = local radius of curvature of the jet
s = distance along line BC
t = thickness of the jet at exit
U = velocity at any point in the jet
x, y = Cartesian coordinates
θ = angle between the velocity vector and the freestream direction at any point in the stream
ρ = density of air
σ = distance normal to local flow direction of jet
τ = jet exit angle
φ = velocity potential
ψ = stream function
ν = angle defined by Eq. (14)

Subscripts

j = jet at ambient pressure
l = local
1 = along line BC
∞ = stream at infinity

I. Introduction

WITH the advent of V/STOL aircraft there will appear an ever-increasing set of fluid dynamics problems involving jets with one total head being injected into an external flow with another total head. In the following pages a theory is developed for the simplest possible of these flows; namely, that caused by the injection of a two-dimensional thin jet at an arbitrary angle into a stream flowing along a plane surface of infinite extent.

For the special case when the injection angle is 180° (see Fig. 1a), the flow may be thought of as being due to that of a translating two-dimensional lift jet impinging on the ground; for instance, during takeoff or landing. One branch of the impinging jet flows forward in the direction of translation and is turned up and away from the ground.[1] It may also correspond to the flow from a two-dimensional thrust reverser with $\tau = 180°$ (Fig. 1b). At any other injection angle the flowfield is that due to a jet exiting from a large plane surface and interacting with a flow from infinity (Figs. 1b and 1c).

The basic approach employed here, involving the use of natural coordinates, was introduced in Ref. 2. The authors of this reference found that the solution of the general problem was far too difficult to be attempted. Instead, an outline was given of possible solutions for the two limiting cases of small and large differences of total head between the jet and the stream. No results of calculations were given. More recent work along the same lines is reported in Refs. 3 and 4. Reference 4 presents results of a numerical procedure for solving the nonlinear potential problem in the velocity plane.

Reference 5 presents a somewhat related study for the case when the injection angle is 180°. However, it is not clear how a jet and a stream of different total heads can exhibit the type of stagnation-point behavior assumed in this reference. In Ref. 6 a rough theoretical analysis for finding the interface shape is given. A semiempirical method is presented in Ref. 7. One of the graphs in this latter reference shows the only test data apparently available for the shape of the jet in two-dimensional flow.

II. Theory

Consider the steady irrotational flow along an infinite plane of an inviscid incompressible fluid of freestream velocity U_∞, which is being disturbed by the injection into the stream of a thin two-dimensional jet with a higher total head at an angle τ with the freestream direction (Fig. 1).

Let us first consider the stream flow. Natural coordinates, i.e., the flow potential φ and the stream function ψ, will be used. In this case, both the flow deflection angle θ and the logarithm of the velocity magnitude q must satisfy Laplace's equation as follows:

$$\theta_{\varphi\varphi} + \theta_{\psi\psi} = 0$$
$$[\ln(q/U_\infty)]_{\varphi\varphi} + [\ln(q/U_\infty)]_{\psi\psi} = 0 \quad (1)$$

Here the subscripts indicate partial differentiation.

Let us arbitrarily assign the value U_j^2t/U_∞, where t is the jet thickness, to the constant potential line passing through the stagnation point (point B, Fig. 1), and let $-\infty$ and $+\infty$ be the values of the potential at infinity upstream and down-

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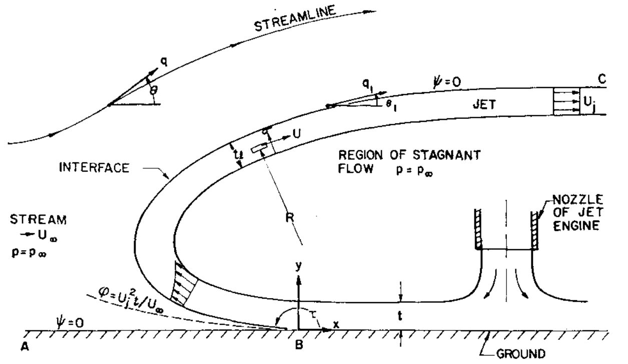


Fig. 1a Flow model of a two-dimensional impinging jet at forward speed ($\tau = 180°$).

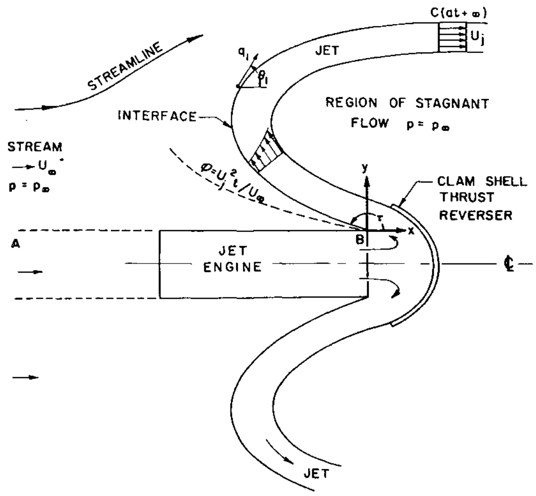


Fig. 1b Flow model of a two-dimensional thrust reverser.

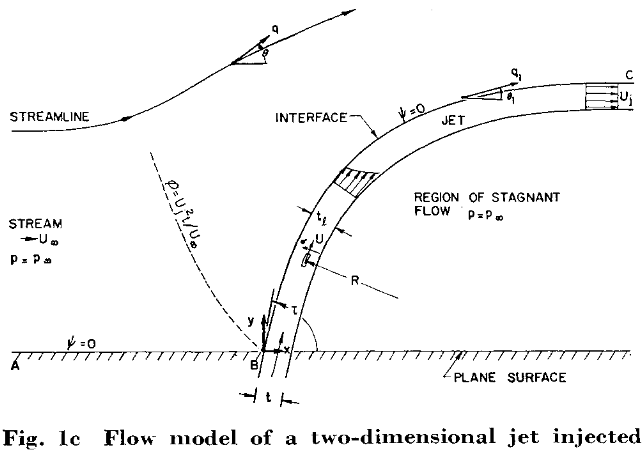


Fig. 1c Flow model of a two-dimensional jet injected into a stream.

stream, respectively. The boundary of the stream is taken to be the line $\psi = 0$.

The following boundary conditions apply:

$$\begin{aligned}\theta(\varphi, 0) &= 0 \qquad \text{for } \varphi < U_j{}^2t/U_\infty \\ &= \theta_1(\varphi) \qquad \varphi > U_j{}^2t/U_\infty\end{aligned} \tag{2}$$

It is known[2] that an expression that satisfies the Laplace equation along the line $\psi = 0$ is given by

$$\ln\left(\frac{q}{U_\infty}\right) = -\frac{1}{\pi}\int_{U_j{}^2t/U_\infty}^{\infty}\frac{\theta_1(\bar{\varphi})d\bar{\varphi}}{\bar{\varphi} - \varphi} \tag{3}$$

Turning next to a consideration of the flow inside the jet, we shall make use of some of the concepts of the so-called "exponential theory" of peripheral jets[8] in order to obtain an expression for the pressure difference across the jet. The pressure in the stagnant (cavity) region is assumed to be the freestream static pressure p_∞. The simplified force balance equation along a direction σ normal to the jet flow, with R as the radius of curvature, is

$$dp/d\sigma = \rho U^2/R \tag{4}$$

Here U is the velocity at an arbitrary point in the jet where the pressure is p. The density ρ is assumed constant. Furthermore, we shall assume that R is constant across the jet (thin jet assumption), but that it varies along the jet. Integration of Eq. (4) across the local jet thickness t_l, after substitution for U from the Bernoulli equation, results in

$$p_l - p_\infty = \tfrac{1}{2}\rho U_j{}^2(1 - e^{-2t_l/R}) \tag{5}$$

Here p_l is the local static pressure at the interface between the stream and the jet. The jet velocity corresponding to ambient pressure is denoted by U_j. If we now approximate by letting $t_l = t$, the thickness of the jet at the exit, and also assume that $t \ll R$, we may expand Eq. (5) in a series. Retaining the first term only, we have

$$p_l - p_\infty = \rho U_j{}^2t/R \tag{6}$$

The radius of curvature of the jet is also the radius of curvature of the stream along the interface BC. Hence, we may write (with s being the distance along the line BC)

$$1/R \doteq -d\theta_1/ds = -q_1(d\theta_1/d\varphi) \tag{7}$$

Note that in general (see Fig. 1c) R is finite at the stagnation point. Since $q_1 = 0$ here, Eq. (7) requires that

$$d\theta_1/d\varphi = -\infty \qquad \text{for } \varphi = U_j{}^2t/U_\infty \tag{8}$$

Introducing the nondimensional parameter ξ, defined by $\xi = U_\infty\varphi/U_j{}^2t$, and using the Bernoulli equation, Eqs. (6) and (7) may be combined to yield

$$-q_1/U_\infty + (q_1/U_\infty)^{-1} = -2(d\theta_1/d\xi) \tag{9}$$

Substituting for (q_1/U_∞) from Eq. (3), we find

$$\frac{1}{\pi}\int_1^\infty \frac{\theta_1(\bar{\xi})d\bar{\xi}}{\bar{\xi} - \xi} = \sinh^{-1}\left[-\frac{d\theta_1}{d\xi}\right] \tag{10}$$

This is a nonlinear singular integral equation, which must hold along the line BC. It is recognized that $d\theta_1/d\xi = 0$ at point C, since the radius of curvature is infinite, whereas $q = U_\infty$ here. Postulating that $d\theta_1/d\xi$ remains small over most of the length BC [we know, however, from Eq. (8) that this is not the case in the immediate neighborhood of point B], Eq. (10) is linearized by expanding the right-hand side in a series and keeping the first term only, i.e.,

$$\frac{1}{\pi}\int_1^\infty \frac{\theta_1(\bar{\xi})d\bar{\xi}}{\bar{\xi} - \xi} = -\frac{d\theta_1}{d\xi} \tag{11}$$

To obtain an easier integration it is convenient to invert Eq. (11) by applying the "Hilbert" transformation.[9] We find

$$(\xi - 1)^{1/2}\theta_1 = -\frac{\xi}{\pi}\int_1^\infty \frac{(\bar{\xi} - 1)^{1/2}(d\theta_1/d\bar{\xi})d\bar{\xi}}{\bar{\xi}(\xi - \bar{\xi})} + \frac{1}{\pi}\int_1^\infty \frac{\theta_1 d\bar{\xi}}{\bar{\xi}} \tag{12}$$

The value of the integral

$$\int_1^\infty \frac{\theta_1 d\bar{\xi}}{\bar{\xi}}$$

is determined by noting that θ_1 is finite when $\xi = 1$. Thus

Eq. (12) becomes

$$\theta_1 = -\frac{(\xi - 1)^{1/2}}{\pi}\int_1^\infty \frac{(d\theta_1/d\bar{\xi})d\bar{\xi}}{(\bar{\xi} - 1)^{1/2}(\xi - \bar{\xi})} \qquad (13)$$

It may be seen from Eqs. (3) and (11) that $d\theta_1/d\xi$ has a logarithmic singularity at the stagnation point. To solve the preceding equation, we define

$$\xi = 2/(1 + \cos\nu) \qquad \text{with } 0 \leq \nu \leq \pi \qquad (14)$$

and assume θ_1 to be given by the following series expansion, which satisfies the logarithmic singularity condition for $d\theta_1/d\xi$ at $\xi = 1(\nu = \pi)$:

$$\theta_1 = \sum_{n=1}^{\infty} A_n[\cos n\nu - (-1)^n] + B\left[\frac{1 + \cos\nu}{2} + \frac{\sin^2\nu}{4}\ln\left(\frac{1 - \cos\nu}{1 + \cos\nu}\right)\right] \qquad (15)$$

with

$$\sum_{n=1,3,5}^{\infty} A_n = \frac{\tau - B}{2} \qquad (15a)$$

Here B and A_n are constants. Equation (15) satisfies, as it should, the following end point conditions:

$$\theta_1 = \tau \qquad d\theta_1/d\xi = -\infty \qquad \text{for } \xi = 1$$

$$\theta = 0 \qquad d\theta_1/d\xi = 0 \qquad \text{for } \xi = \infty$$

The substitution of Eq. (14) into Eq. (13) yields

$$\theta_1 = -\frac{\sin\nu}{\pi}\int_0^\pi \frac{(1 + \cos\bar{\nu})^2(d\theta_1/d\bar{\nu})d\bar{\nu}}{\sin\bar{\nu}(\cos\bar{\nu} - \cos\nu)} \qquad (13a)$$

By contour integration of the preceding equation, after substitution for θ_1 from Eq. (15), we find

$$-\left\{\sum_{n=1}^{\infty} A_n[\cos n\nu - (-1)^n] + B\left[\frac{1 + \cos\nu}{2} + \frac{\sin^2\nu}{4}\ln\left(\frac{1 - \cos\nu}{1 + \cos\nu}\right)\right]\right\} = -A_1\left(\sin\nu + \frac{1}{4}\sin 2\nu\right) - A_2\left(\frac{7}{2}\sin\nu + 2\sin 2\nu + \frac{1}{2}\sin 3\nu\right) - \sum_{n=3}^{\infty} nA_n\left\{\sum_{1}^{n-2}\sin n\nu + \frac{7}{4}\sin(n-1)\nu + \sin n\nu + \frac{1}{4}\sin(n+1)\nu\right\} + B\left\{\frac{\pi}{4}[(1 + \cos\nu)^2 - (1 + \cos\nu)^3] - \frac{1}{2}\sin\nu(2 + \cos\nu)\right\} \qquad (16)$$

The value of B is next determined from this equation by setting $\nu = 0$, i.e.,

$$B = \tau/\pi \qquad (17)$$

If the previous, infinite sums are truncated at a given n, we have a functional relation between n unknown coefficients $A_1, A_2, \ldots A_n$, subject to the constraint equation, Eq. (15a). We could now determine the unknown A's by generating $n - 1$ equations at $n - 1$ values of ν and using Eq. (15a) to get the nth equation. However, the convergence of such a collocation method is not expected to be good. Instead, we may satisfy the jet-stream interface condition [Eq. (16)] by employing the method of least squares, subject to a constraint using Lagrange's method of undetermined multipliers. Specifically, the interval of ν from 0 to π is divided into $\mu + 1$ equal intervals of width $\Delta\nu$. Defining F as the difference of the sum of terms on the left-hand side and the sum of terms on the right-hand side of Eq. (16), a new function f is formed, as follows:

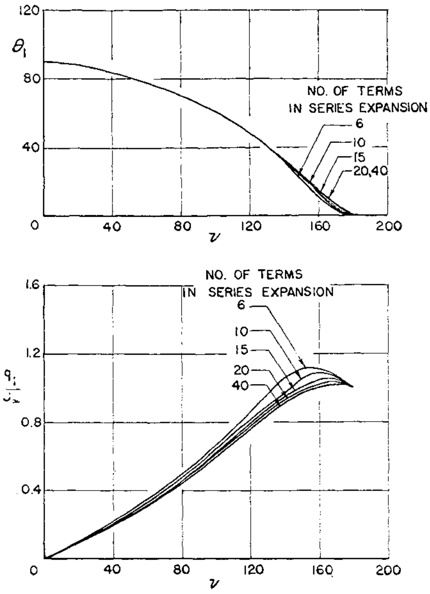


Fig. 2 Solution as a function of number of terms used in series expansion of θ_1 ($\tau = 90°$). $U_\infty/U_j = 0.6$.

$$f = \sum_{\nu=\Delta\nu}^{\nu=\mu\Delta\nu} F^2(\nu) + \lambda\left\{\sum_{n=1,3,5} A_n - \frac{\tau}{2}\left[1 - \frac{1}{\pi}\right]\right\} \qquad (18)$$

The function f is a measure of how well the linearized boundary condition [Eq. (16)] is satisfied along the line BC. Differentiation of f with respect to the unknown A_n's and λ leads to a set of $n + 1$ simultaneous linear algebraic equations in $n + 1$ unknowns.

With the coefficients in the expansion known, the flow deflection angles θ_1 at each point ν along the interface BC are known from Eq. (15). The magnitudes of the velocity along the ground and along the interface can be obtained by performing the integration indicated in Eq. (3). The velocity along the ground will not be presented here, however.

The x and y coordinates corresponding to each value of ν along the interface may be calculated from the expressions given below (obtained from $ds = d\varphi/q_1$):

$$\frac{x}{t}\left[\frac{U_\infty}{U_j}\right]^2 = 2\int_0^\nu \frac{\sin\nu\cos\theta_1\, d\nu}{(1 + \cos\nu)^2 q_1/U_\infty} \qquad (19)$$

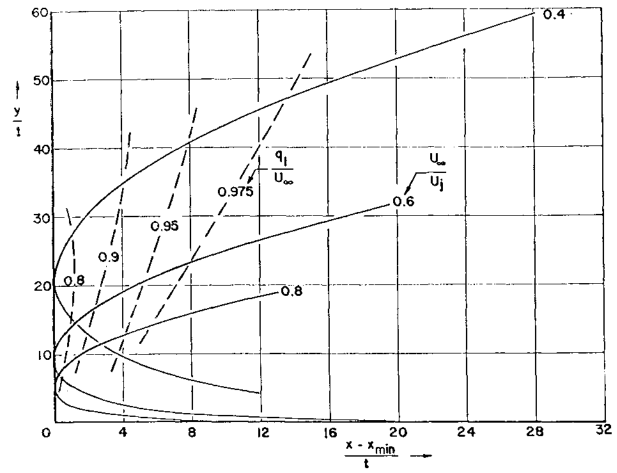
$$\frac{y}{t}\left[\frac{U_\infty}{U_j}\right]^2 = 2\int_0^\nu \frac{\sin\nu\sin\theta_1\, d\nu}{(1 + \cos\nu)^2 q_1/U_\infty} \qquad (20)$$

Here θ_1 and q_1/U_∞ are given by Eqs. (15) and (3), respectively. The shape of the interface is therefore determined through this numerical integration.

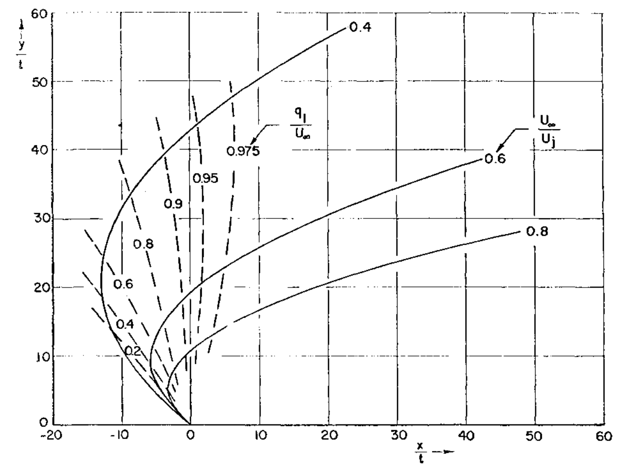
III. Discussion

The method developed in the preceding pages was programmed on the CDC-3600 digital computer at the University of California, San Diego. The results obtained indicated a somewhat low level of convergence of the method unless a large number of terms in the series expansion were taken. By running the program with a steadily increasing number of terms in the series for each combination of the independent variables τ and U_∞/U_j, limiting curves seemed to be reached beyond which further increases in the number of terms gave no change in the shape of the interface in the region of plotting. Typical curves are presented for $\tau = 90°$ in Fig. 2. The final curves (Figs. 3a–3c) were obtained with 40 terms in the series expansion and required approximately 10 min total running time on the CDC-3600 computer.

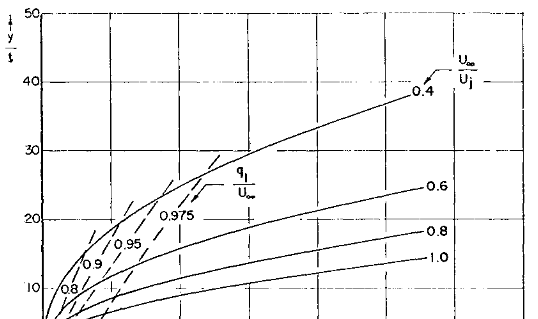
It might be mentioned that the flow model used yields a wake that is infinite in width far downstream, i.e., $y = \infty$ when $x = \infty$. This is, of course, the same result as that obtained for the free streamline issuing from a flat plate mounted



a) Jet exit angle τ = 180°



b) Jet exit angle τ = 135°



c) Jet exit angle = τ90°

Fig. 3 Location of and velocity along jet-stream interface.

perpendicular to an oncoming stream, when the "wake" pressure is assumed to be ambient.

In the limiting case when τ = 180° (i.e., the jet is opposing the stream), it was found that the jet leaves the ground at infinity. The origin of our axis system is then at infinity. To allow plotting, the shapes of the interface for τ = 180° were therefore displaced horizontally as shown in Fig. 3a so that their points of maximum penetration into the stream are located on the y axis. It is noted that the jet interface becomes wider as the freestream velocity is decreased. In all cases presented in this report it was found that the velocity magnitude q_1 along the interface was a monotonic increasing function of the distance s along the interface.

The calculated results for τ = 135° and 90° are shown in Figs. 3b and 3c, respectively. The curves indicate a pre-

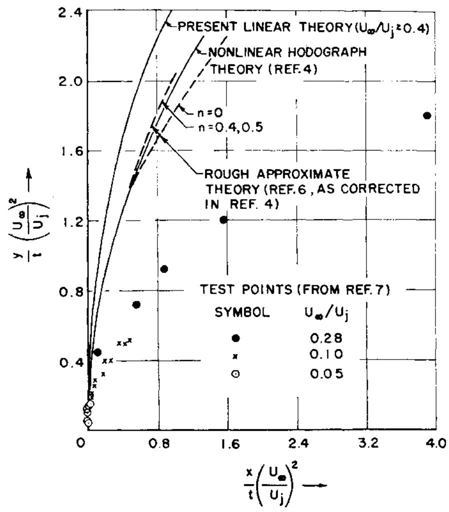


Fig. 4 Jet shape, comparison between theoretical and experimental results, τ = 90°.

dictable trend in that the penetration of the jet into the stream becomes more pronounced as the freestream velocity decreases. The dotted curves show how the local velocity increases along the interface to reach the value of U_∞ at infinity downstream.

In linearizing Eq. (10) it was assumed that the parameter $(d\theta_1/d\xi)$ remains small over most of the length BC. Having now obtained the linearized solution, it is pertinent to go back and check the validity of this assumption by numerically calculating this parameter along the interface. This was done for the case when τ = 90°. The results indicate that the linearizing assumption is valid for $U_\infty/U_j \geq 0.4$ (approximately).

The only test data available so far are presented in Fig. 6. Note that the test points are for values of $U_\infty/U_j \leq 0.28$ and are therefore not strictly applicable for comparison with the present linear theory ($U_\infty/U_j \geq 0.40$). A comparison of the results of the present linear theory for τ = 90° with other theoretical data is also shown on Fig. 4. Taylor[6] mentions that the jet, due to viscous mixing, would fill a wedge of nearly 40°. In view of this, Ackerberg and Pal[4] state that the discrepancy between theory and experiment might not be so great.

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